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THE ABSORPTION AND SCATTERING OF RADIATION IN RANDOM AGGREGATES OF PEBBLES

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THE ABSORPTION AND SCATTERING OF RADIATION IN RANDOM AGGREGATES OF PEBBLES

by

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Abstract

The attenuation of a collimated mono-energetic beam of radiation in traversing random aggregate of pebbles is considered. It is found that there is a transmission of radiation over and above that allowed by the mean density of material and that transmission is essentially exponential. This implies that a shield composed of pebbles must be increased in depth by a given ratio over that computed by means of the average density alone. It is shown that this ratio, called here the channelling effect factor, f, is given with fair approximation by the formula

$$f = 1 + 0.54\,\lambda\tau(1 - \rho)$$
$$(\lambda\tau \ll 1)$$

where λ is the reciprocal of the average of the reciprocal of the mean solid and void segmental lengths, τ the linear absorption coefficient in the solid material, and ρ, called the "reduced density" is the fraction of volume occupied by solid material.

The problem of computing the channelling effect reduces itself to the problem of determining the probability of penetrating an accumulated depth x' of solid material in traversing a total distance x in the pebble aggregate. The theory for this computation is developed in general for cases of continuous distribution of solid and void segments; and for cases where these distributions have only discrete values. Assuming constant the probability of a segment ending in an elemental length for each of the two viewpoints results in formulae unlike in appearance, but which give essentially the same numerical result when the parameters are interpreted properly. The theory for the discrete distributions is de-

veloped with the size of the discrete sub-unit of length as a parameter. It is shown that a particularly simple result occurs when the reciprocal of the length of the sub-unit is equal to the sums of the reciprocals of the average void and average chord lengths. It is further shown that the results for the discrete case go into the results of the continuous case as the length of the discrete sub-unit is allowed to go to zero.

The effect applies not only to attenuation, but also to diffusion; that is, it is a purely geometrical effect. This follows from the fact that the probability of scattering or absorption is found by weighting the probable amount of solid penetrated by a negative exponential, that is, by the appropriate transport diffusion kernel. Therefore the discussion applies not only to shielding but to all problems dealing with the transmission and diffusion of radiation through a bed of randomly oriented pebbles, and is of sufficient magnitude to warrant attention whenever the pebble size is comparable to the relaxation length.

Whenever the interstices between the pebbles are filled the channelling effect still exists, but in reduced amount. A method of estimating the effect for this case is given.

Introduction

One of the problems in nuclear reactor technology is that of getting a gas coolant into and out of reactor through the shield, and at the same time preserving the shielding and, in some applications, keeping the weight to a minimum. The various proposals for accomplishing this include the shielded labarinth (the ORNL graphite reactor is an example), a porous shield composed of a medium in which many holes with incommensurate wiggles are placed (the "perforated shield"), and porous shield composed of a random stack of appropriately shaped pebbles to cut down flow resistance (the pebble shield).[1]

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[1]G. Gamow and Nicholas M. Smith Jr., Project RAND Symposium of August 1947, reported in their RAD-198.

Considered herein is the geometrical effect of the voided volumes on the attenuation (and scattering) of a random stack of pebbles. By "random stack" is meant that the distribution of the material is statistically isotropic. It is assumed that randomness is achieved by dumping the pebbles en masse. If the pebbles form regular lattice arrays the stack will not be "random".

One's first thought is that the attenuation in a random stack of pebbles would be that of a homogeneous medium of the same material and of the same average density as the pebble stack (referred to here as the "density effect"). There is, however, a net transmission caused by the presence of voids in a pebble stack which is over and above the transmission allowed by the average density alone.[1]

This added transmission (called herein the "channelling effect") is allowed by virtue of the statistical fluctuation from ray to ray in solid material penetrated. The average of this quantity is, of course, just that corresponding to the average density. But since the probability of transmission of radiation along a ray is weighted by a negative exponential factor, the thin spots therefore always transmit more radiation than is removed by the thick spots. This effect depends on there being a distribution of solid path lengths through the shield. If the solid material penetrated is the same for every parallel ray (e.g., the case of parallel slabs of shield with voids between) the distribution is nil, i.e., is a delta function, and the transmission is that predicted on the basis of average density alone. It shall be demonstrated that the pebble shield must be increased in thickness by a given ratio (for the same attenuation as a homogeneous shield of the same density) because of the channelling effect. This ratio, called herein " the channelling effect factor " becomes appreciable as the average void size becomes of comparable magnitude to the relaxation length in the solid material. For instance, a good rule of thumb is that the channelling effect factor is given by $f = 1 + .54\ \lambda\tau(1-\rho)$ where $\lambda = (\overline{S}^{-1} + \overline{V}^{-1})^{-1}$, $\overline{S}$ is the average solid continuous path length, $\overline{V}$ the average void continuous path length, and ρ the "reduced density", i.e., the fraction of total volume which is filled by solid material, and τ is the linear absorption coefficient in the solid material.

The problem, then, reduces itself to that of computing the probability of penetrating a thickness x' of solid material in traversing a total distance x in the pebble bed. This probability distribution will depend on the detailed nature of the distribution of the continuous solid and void rays (or segments) determined by straight lines through the pebble stack. These segmental distributions in general are complicated; that of the solid segments can be computed analytically for a few regular shapes of pebbles; however, that of the voids involves a concise knowledge of the manner in which the pebbles stack. The problem shall be approached in a more Aristotelian manner in that simple functions for the distribution of solid and void lengths will be assumed which will allow a manageable analytical analysis of the problem. The theory will contain two adjustable parameters which will allow, in effect, the matching of the first and second moments (i.e. the average amount of solid material penetrated and the standard deviation from the average amount of solid material penetrated) of the distribution giving the probability of encountering x' amount of solid material in penetrating the bed a distance x. Since the first two moments can be matched, fairly good estimates of the channelling effect can be expected; and in particular it can be expected that an empirical determination of the adjustable parameters for one special case (one experiment with one shape of pebble) will allow one to calculate the channelling effect for all cases involving that particular shape of pebble, i.e. for all sizes and values of the absorption coefficient.

2. General Theory (Continuous Distributions)

One desires to know the probability that in a given ray of length x in a stack of pebbles a total solid thickness x' of material is traversed. The pebble bed is assumed to have statistically isotropic properties.

Let $S(n,x)dx$ = the probability that if n solid segments are selected at random from the same distribution that the sum of the n lengths lies between x and x + dx. The variable n has integral values from one to ∞.

Let $V(n,x)dx$ = the corresponding distribution for the void segments.

Centering attention on $S(1,x)$, the probability distribution of the length of one segment, it is noted that $S(n,x)$ can be formed from $S(1,x)$ by an $(n-1)$-fold convolution of $S(1,x)$ with itself. This leads to the recursion formula:

$$S(n,x) = \int_0^x S(n-1, x-x') S(1,x')dx' \quad . \qquad (1)$$

This formula can be explained by dividing the interval from 0 to x into two at x'. Then $S(1,x')dx'$ is the probability that one segment fills the interval from 0 to x' ending in dx', and $S(n-1, x-x')$ is the probability that n-1 segments fill the remaining interval from x' to x.

In a similar manner the functions $V(n,x)$ can be obtained by an (n-1) fold convolution of $V(1,x)$ on itself.

Now in traversing a given ray through a pebble bed, one encounters alternate solid and void segments whose distributions are given by $S(1,x)$ and $V(1,x)$ respectively. In any one ray n solid segments may be encountered, and n, or $n \pm 1$ void segments may be encountered. That is, there is an equal number of void segments to solid segments, or one more or less.

In principle the n solid segments from the $S(1,x)$ distribution may be assembled together in the interval from 0 to x', and the remaining n, or $n \pm 1$ void segments assembled in the interval from x' to x (See Figure 1).

Thus, instead of making alternate random selections from first one, then the other of the $S(1,x)$ or $V(1,x)$ distributions we may make first one selection from the $S(n,x)$ distribution, then one from the $V(n,x)$ [or $V(n+1,x)$. or $V(n-1,x)$] distribution. We now consider all the ways in which the interval from 0 to x' can be filled with solid segments and the interval from x' to x with void segments. Since x' is located between 0 and x, there must be at least one $S(1,x)$ segment in the distribution. There can be, however, one more solid than void, vice versa, or equal numbers. This gives rise to three infinite sums over n.

Figure 1

A sample array of segments illustrating the manner in which a simple convolution of the built-up functions $S(n+1,x)$ and $V(n,x)$ -- for the case of starting and ending with a solid segment -- is used to replace the 2n-fold convolution of alternate solid and void segments. Without loss of generality all of the solid segments may be assembled together on the interval between zero and x' ; all of the void segments on the interval between x' and x. The n+1 solid segments may then be replaced by one segment from the $S(n+1,x)$ distribution, which gives the probability that n+1 segments from the $S(1,x)$ distribution sum up in length in the interval dx around x. An analogous statement applies to the replacement of the n void segments by one segment from the $V(n,x)$ distribution. The product of $S(n+1,x')$ and $V(n, x-x')$ is then summed for all permissible values of n to give the probability that an array which begins and ends with a solid segment contains a length x' of solid material. This sum must be multiplied by the probability of starting the array with a solid segment, and then divided by the probability that the last solid segment ends in the element dx at x, since this last restriction -- already implicit in the definition of the $S(1,x)$ distribution -- must be removed as one is unconcerned with what follows after the end of the array. Other sums, properly weighted, are formed for arrays which start with solid material and end in void, etc.

A SAMPLE ARRAY OF SEGMENTS

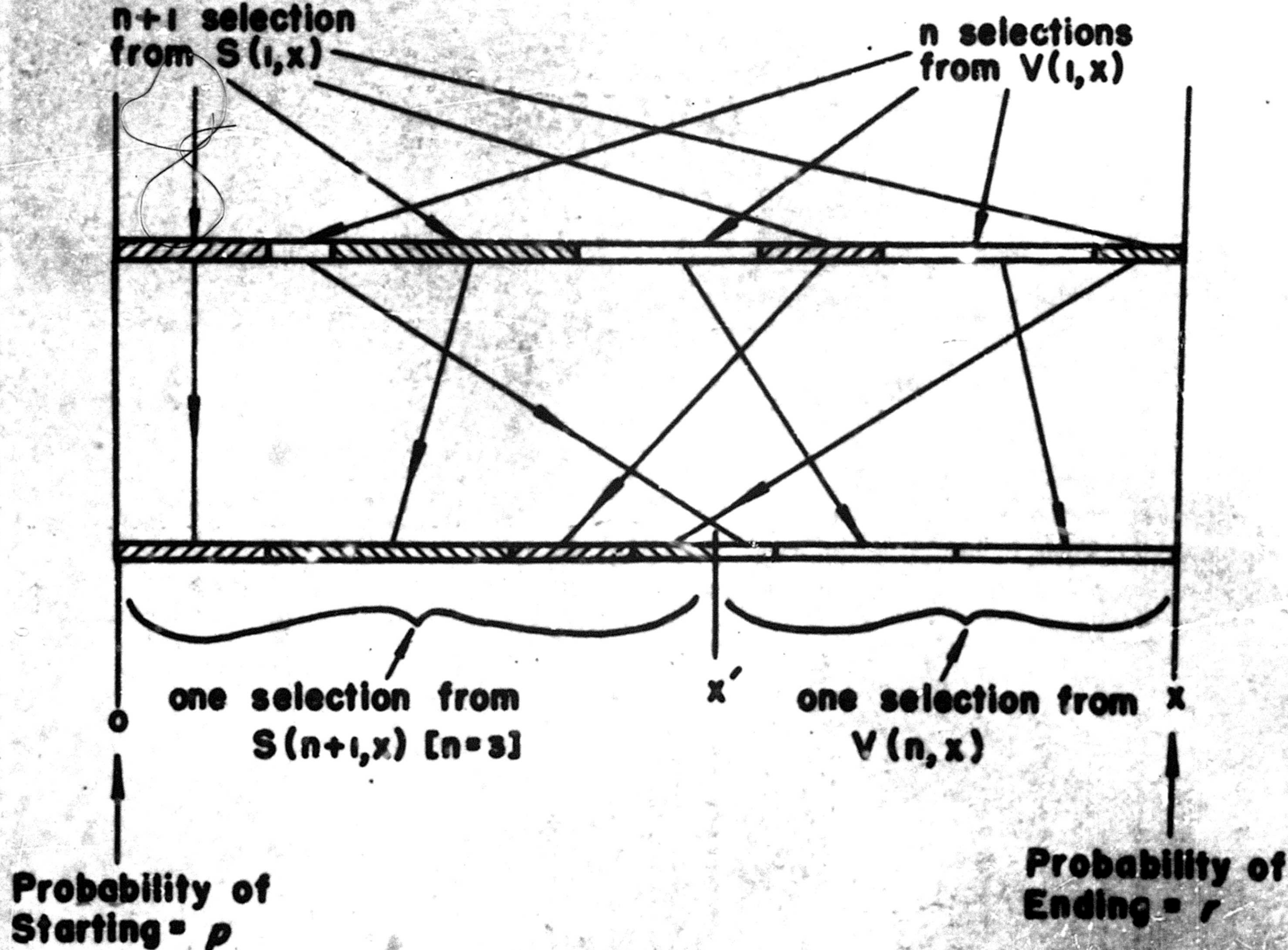

FIG. 1

The probability that there shall be x' total length of solid matter and that the sum of the solid and void lengths shall equal x is then:

$$F(x',x) = (q/r)\; S(1,x)\,\delta(x-x')$$
$$+ (q/r)\sum_{n=1}^{\infty} S(n+1,x')\; V(n,x-x')$$
$$+ (q/s + p/r)\sum_{n=1}^{\infty} S(n,x')\; V(n,x-x')$$
$$+ (p/s)\sum_{n=1}^{\infty} S(n,x')\; V(n+1,\; x-x')$$
$$+ (p/s)\, V(1,x)\;\delta(x'-0).$$

The summations are respectively for one more solid than void, equal number of solid and void, and one more void than solid segment. For completeness, when x' = x we can have the case of one solid and no void segments; and when x' = 0 we can have the case of no solid and one void segments. These extra terms are contained in the values including the delta functions.

The weighting factors, p, q, r, and s are defined as follows:

p = the probability of starting with a void segment in dx at x=0,
q = the probability of starting with a solid segment in dx at x=0,
r = the probability of a solid segment ending in dx at x, and
s = the probability of a void segment ending in dx at x.

Consider the terms corresponding to an array which has one more solid segment than void segment, i.e. it starts with a solid segment and ends with a solid segment. These terms are weighted with q as a multiplier to allow for the chance of starting with a solid segment. At x, the end point of the array, the final solid

segment so far is constrained to end in dx at x. Since this constraint is undesirable we remove it by dividing by r, the probability of solid segment ending in dx at x. (It will be remembered that S(1,x) is the probability for a solid segment to extend for a length x and then end in dx at x.) For those terms corresponding to arrays starting with solid and ending with void segments the weighting factor is q/s; for those starting with void and ending with solid segments the factor is p/r; and for those starting with void and ending with void segments the factor is p/s. Letting $\bar{S}$ represent the average value of x in the S (1,x) distribution; and similarly $\bar{V}$ represent the average value of x in the V(1,x) distribution; that is:

$$\bar{S} = \int_0^{\infty} x\, S(1,x)\,dx, \text{ and}$$
$$\bar{V} = \int_0^{\infty} x\, V(1,x)\,dx, \tag{3}$$

a moments reflection will show that

$$p = \frac{\bar{V}}{\bar{S}+\bar{V}}\,dx\;, \qquad q = \frac{\bar{S}}{\bar{S}+\bar{V}}\,dx,$$
$$r = \frac{dx}{\bar{S}}\;, \qquad s = \frac{dx}{\bar{V}}\;, \tag{4}$$

since the pebble bed is statistically isotropic.

Since all distributions used have been normalized, and all sums properly weighted we expect that the integral

$$\int_0^\infty F(x',x)\, dx' = 1. \qquad (5)$$

Equation 5 will serve as a valuable check when explicit functions are introduced for the S and V functions.

Letting τ represent the absorption (or scattering) probability per unit length of the solid material, the reduction in intensity of the primary radiation in penetrating a distance x of the pebble bed is given by

$$\frac{I(x)}{I(o)} = \int_0^\infty e^{-\tau x'} F(x',x).\, dx', \qquad (6)$$

It should be pointed out that the general theory contains implicitly one assumption which may or may not be true. Namely, that in the selection of a sequence of segments, alternately void and solid, that each selection can be made independently of the preceding selections. As our distribution contains logically an infinite number of segments, the depletion caused by the removal of any one segment does not alter the distribution.

But physically, if one pebble has assumed a particular orientation, this orientation may alter the orientation of the next pebble. However, even though it may be true that no sequence of lengths in a given array is random in itself; it becomes random by losing its identity in the total ensemble of arrays.

3. A Particular Case - Exponential Distribution of Solids and Voids.

In order to get functions which can be handled analytically and which have some similarity to reality, the assumption is made that the probability of a solid or void segment ending in the infinitesimal dx is uniform - i.e., the ending of the segment occurs at random. This assumption can be easily shown to be equivalent to the assumption that the distribution functions, S(1,x) and V(1,x) are exponentials with negative powers. These distributions could be further altered to chop off the long segments, or otherwise, but in the absence of knowledge of the form of both S and V, and the desire for analytically usable functions, it is sensible to stay with the completely exponential distribution. F. H. Murray[2] has computed the distribution of chord lengths for the right circular cylinder and cube and this distribution is certainly not exponential, some values of the longer chords being much more probable than the very short or the longest chords.

When the pebbles have flat sides, there is an appreciable probability that one flat side is partially against another. In this manner chord lengths can be built up which are longer than the longest chord inside one pebble.

The distributions assumed in this section would correspond more nearly to a combination of crushed and unscreened absorbing and non-absorbing material, such as iron and graphite. Nevertheless the first moments of the distributions may be matched to the experimental case, and the second moments will be of the same order of magnitude. Hence, although non-similar in appearance, the end result will not be expected to diverge too widely from the actual case. The theory is in effect built on an arbitrary model but supplied with two adjustable parameters.

Furthermore, the theory at the very least will serve to guide experimentation, to show the qualitative dependence on the parameters involved and may be used in comparison with other approximate methods of evaluation. It is possible that there exists

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other simple functions for S(1,x) and V(1,x) which are better approximations and which also yield integrable functions. We assume then that

$$S(1,x) = u\, e^{-ux}$$
$$V(1,x) = v\, e^{-vx} \qquad (7)$$

We note at once that

$$\bar{S} = \frac{1}{u},$$
$$\bar{V} = \frac{1}{v},$$
$$p = \frac{v}{u+v}\, dx, \qquad (8)$$
$$q = \frac{v}{u+v}\, dx,$$
$$r = u\, dx,$$
$$s = v\, dx.$$

Furthermore, we can prove by simple induction on n that

$$S(n,x) = u\,\frac{(ux)^{n-1}\, e^{-ux}}{(n-1)!} \qquad (9a)$$

Formula 9a obviously holds for $n = 1$. Assume it holds for $n - 1$. Then the integration of equation 1 proves it also holds for n, completing the induction. By similar process we have

$$V(n,x) = v\,\frac{(vx)^{n-1}\, e^{-vx}}{(n-1)!} \qquad (9b)$$

Substituting equation 9 in equation 2 one obtains:

$$F(x'x) = \frac{v}{u+v}\, e^{-ux}\ \delta(x-x')$$
$$+ \frac{v^2}{u+v}\, e^{-[ux' + v(x-x')]} \sum_{n=1}^{\infty} \frac{(ux')^n}{n!}\, \frac{(v[x-x'])^{n-1}}{(n-1)!}$$
$$+ \frac{2uv}{u+v}\, e^{-[ux' + v(x-x')]} \sum_{n=1}^{\infty} \frac{[uv\, x'(x-x')]^{n-1}}{[(n-1)!]^2}$$
$$+ \frac{u^2}{u+v}\, e^{-[ux' + v(x-x')]} \sum_{n=1}^{\infty} \frac{(ux')^{n-1}}{(n-1)!}\, \frac{[v(x-x')]^n}{n!}$$
$$+ \frac{u}{u+v}\, e^{-vx}\ \delta(x'-0) \qquad (10)$$

By means of the definitions of the Bessel function of imaginary argument[3]:

$$J_p(2i\sqrt{t}) = (i\sqrt{t})^p \sum_{m=0}^{\infty} \frac{t^m}{m!\,(m+p)!}, \qquad (11)$$

from which the summations of equation 10 may be expressed:

[3] Jahnke-Emde "Tables of Functions," p 128, Dover Publication.

$$F(x',x) = \frac{v}{u+v}\; e^{-ux}\; \delta(x-x') + \frac{u}{u+v}\; e^{-vx}\; \delta(x'-0)$$

$$+\left\{\frac{vx' - u(x-x')}{i\sqrt{uvx'(x-x')}}\; J_1\left(2i\sqrt{uvx'(x-x')}\right) + 2\,J_0\left[2i\sqrt{uvx'(x-x')}\right]\right\}\frac{uv e^{-[ux' + v(x-x')]}}{u+v} \quad . \quad (12)$$

For plotting purposes we note that the argument of the Bessel Functions in equation 12 is of higher values than are tabulated. It is convenient, then, to use an approximation [4]

[4] British Association Mathematical Tables VI, Cambridge.

obtained from the asymptotic series for high values of z:

$$J_0(iz) = \frac{e^z}{\sqrt{2\pi z}}\left\{1 + \frac{1^2}{1!\,8z} + \frac{1^2 . 3^2}{2!\,(8z)^2} + \frac{1^2.3^2.5^2}{3!\,(8z)^3} \ldots\ldots\right\} ;$$

$$iJ_1(iz) = \frac{e^z}{\sqrt{2\pi z}}\left\{1 - \frac{1\quad 3}{1!\,(8z)} - \frac{1^2.3.5}{2!\,(8z)^2} - \frac{1^2.3^3.5.7}{3!\,(8z)^3} \ldots\right\} \quad (13)$$

(all terms for J_1 being negative except the first).

A numerical investigation shows that the first two terms of the series yield values better than 10^{-3} of tabulated values for $z \geq 10$. Hence, using two terms in the expansion and rearranging terms we get:

$$F(x',x) \approx \frac{uv}{2\sqrt{\pi}}\; \frac{e^{-(\sqrt{a} - \sqrt{b})^2}}{g^{3/2}}\left\{(c+d+2g) - \left(\frac{c+d}{g} + \frac{2}{3}\right)\frac{3}{16}\right\}$$

where

$$a = ux'$$

$$b = v(x-x')$$

$$c = vx'$$

$$d = u(x-x')$$

$$g = uvx'(x-x') = ab = cd$$

Using only the first term in the curly brackets, one gets values good to better than 2% for $z > 8$. This expression diverges for $x' = 0$ and $x' = x$; but $F(x',x)$ for practical reasons may be considered zero on these values. (See Fig 2).

Using $x = 20$ cm, $u = 0.3$ cm^{-1}, and $v = 0.6$ cm^{-1}, $F(x'x)$ has been evaluated and plotted in Figure 2, by formula 12.

We now desire to evaluate the integrals

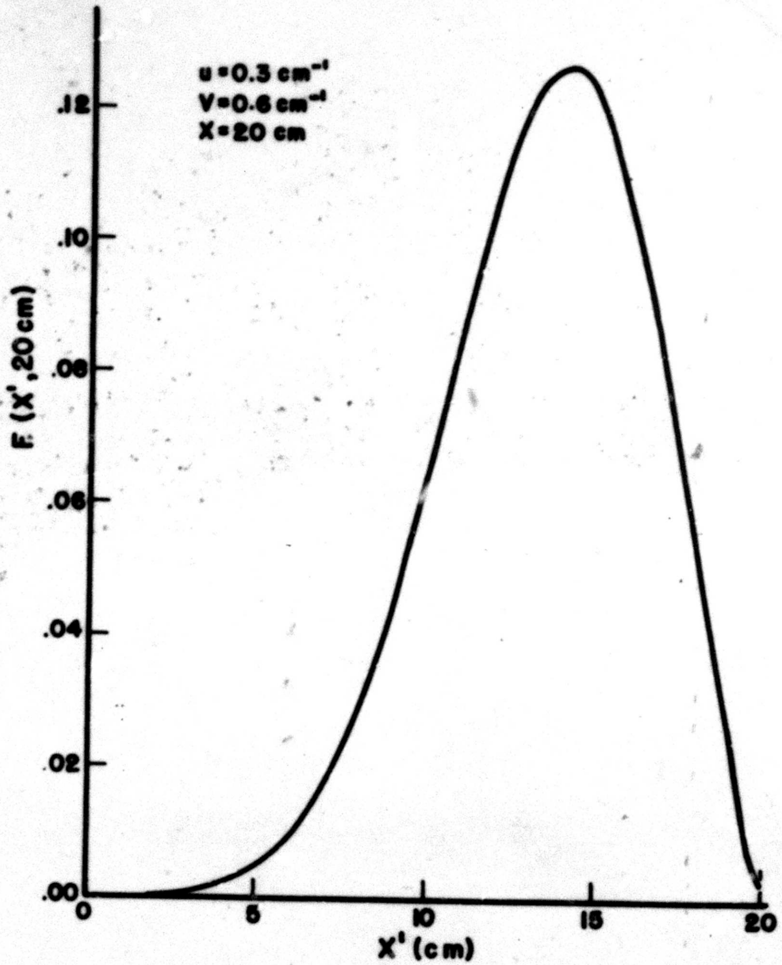
$$\int_0^x F(x',x)\, dx \quad \text{and} \quad \int_0^x e^{-\tau x'}\, F(x',x)\, dx'.$$

Since the first integral may be obtained from the second by allowing $\tau \to 0$ we shall evaluate the second integral. To do so we make use of the Laplace Transform Method.[5]

[5] A good text sufficient for the purposes of this discussion is Churchill, "A Modern Operational Mathematics in Engineering", McGraw-Hill Book Co., Inc., New York, 1944.

The following elementary operations will be performed:

a. The definition of the Laplace transformation of the



-FIG. 2-

THE PROBABILITY OF PENETRATING X' cm OF SOLID MATERIAL IN TRAVERSING 20 cm OF PEBBLE BED HAVING $u = 0.3\ cm^{-1}$ AND $V = 0.6\ cm^{-1}$

function F(x), [throughout this discussion the capital letter will denote a function and the same letter but lower case will denote its Laplace transformation]:

$$L\{F(x)\} = f(z) = \int_0^\infty e^{-zx}\ F(x)\ dx, \qquad (15a)$$

where z is in general a complex number and where the symbol $L\{\ \}$ denotes "the Laplace transformation of",

b. The addition theorem

$$L\{A\ F(x) + B\ G(x)\} = A\ f(z) + B\ g(z), \qquad (15b)$$

where $f(z) = L\{F(x)\}$, $g(z) = L\{G(x)\}$

c. The convolution theorem

$$L\left\{\int_0^x F(t-x)\ G(x)\ dt\right\} = f(z)\ g(z) \qquad (15c)$$

d. The displacement theorem

$$L\{e^{ax}\ F(x)\} = f(z-a) \qquad (15d)$$

e. The inverse operation, denoted $L^{-1}\{\ \}$

$$L^{-1}\{f(z)\} = F(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{xz}\ f(z)\ dz \qquad (15e)$$

The complex integral is evaluated along a path from $\gamma - i\infty$ to $\gamma + i\infty$ with γ of such value that the path is to the right of all poles.

Operations a and e may often be evaluated from tables of Laplace transformation[6].

[6] Ibid. Appendix III, see also Campbell and Foster "Fourier Integrals", D. van Nostrand Co., New York, 1948.

f. The Cauchy residue theorem:

If f(z), a complex function, can be expanded in the neighborhood of a pole z_o in the form

$$f(z) = \sum_{i=m}^{\infty} a_i (z - z_o)^i$$

where $\underline{m}$ is some negative integer and a_i are constants, then

$$\frac{1}{2\pi i} \oint f(z)\, dz = a_{-1}. \qquad (15g)$$

The constant a_{-1} in the expansion is called the residue, and m the order of the pole. If there are more than one pole, then

$$\frac{1}{2\pi i} \oint F(z)\, dz = \Sigma \text{ (residues)}. \qquad (15h)$$

Since equations (1) and (2) are in convolution form we note that (from 15c)

$$\left.\begin{aligned} s(n,z) &= L\{S(n,x)\} = [s(1,z)]^n \\ v(n,z) &= L\{V(n,x)\} = [v(1,z)]^n , \end{aligned}\right\} \qquad (16)$$

and

$$\left.\begin{aligned} s(1,z) &= \int_0^\infty e^{-xz}\, u e^{-ux}\, dx = \frac{u}{z+u} \\ v(1,z) &= \frac{v}{z+v} , \end{aligned}\right\} \qquad (17)$$

[the symbol $\underline{v}$ above represents the parameter; the Laplace transformation will be represented by v(n,x)].

We define a function R(n,x)

$$R(n,x) = e^{-\tau x}\; S(n,x) \qquad (18)$$

and note by equation 15d that

$$r(n,z) = \left[\frac{u}{z+u+\tau}\right]^n$$

The integral $\int_0^x e^{-\tau x'} F(x',x)\, dx'$ may be expressed in convolution form, since

$$\begin{aligned} \int_0^x e^{-\tau x'} F(x',x) &= \frac{v}{u+v}\, e^{-(u+\tau)x} + \frac{u}{u+v}\, e^{-vx} \\ &+ \frac{v}{u(u+v)} \int_0^x \sum_{n=1}^{\infty} R(n+1,x')\, V(n,x-x')\, dx' \\ &+ \frac{2}{u+v} \int_0^x \sum_{n=1}^{\infty} R(n,x')\, V(n,x-x')\, dx' \\ &+ \frac{u}{v(u+v)} \int_0^x \sum_{n=1}^{\infty} R(n,x')\, V(n+1,x-x')\, dx'. \end{aligned} \qquad (20)$$

Let H(x) represent the three last terms of equation 20, then, by equation 15b (the addition theorem) and 15c

$$h(z) = L\{H(x)\} =$$

$$\begin{aligned} &\frac{v}{u(u+v)} \sum_{n=1}^{\infty} r(n+1,z)\; v(n,z) \\ &+ \frac{2}{u+v} \sum_{n=1}^{\infty} r(n,z)\; v(n,z) \\ &+ \frac{u}{v(u+v)} \sum_{n=1}^{\infty} r(n,z)\; v(n+1,z). \end{aligned} \qquad (21)$$

Since $\sum_{n=1}^{\infty} t^n = \frac{t}{1-t}$, $t<1$, we have

$$h(z) = \frac{uv^2}{u+v} \; \frac{1}{(z+u+\tau)(z-z_1)(z-z_2)}$$

$$+ \frac{2uv}{u+v} \; \frac{1}{(z-z_1)(z-z_2)} \qquad (22)$$

$$+ \frac{u^2v}{u+v} \; \frac{1}{(z+v)(z-z_1)(z-z_2)},$$

where z_1 and z_2 are roots of the equation

$$(z+u+\tau)(z+v) - uv = 0;$$

$$z_1 = -\tfrac{1}{2}\left(u+v+\tau - \sqrt{(u+v+\tau)^2 - 4v\tau}\right),$$

$$z_2 = -\tfrac{1}{2}\left(u+v+\tau + \sqrt{(u+v+\tau)^2 - 4v\tau}\right). \qquad (23)$$

The inverse transformation

$$L^{-1}\{h(z)\} = H(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zx}\, h(z)\, dz$$

is elementary in its evaluation, since all the poles of the integrand are simple. The residue of $\oint f(z)\, dz$ where $f(z)$ has a simple pole at $z = z_1$ can be evaluated by the limiting process:

$$\lim_{z\to z_1} (z-z_1)\, f(z) = \text{residue at } z_1$$

By this process (or by tabulated formula #14, page 295, Churchill) one gets that

$$L^{-1}\left\{\frac{1}{(z+a)(z+b)(z+c)}\right\} = \frac{(c-b)\,e^{-at} + (a-c)\,e^{-bt} + (b-a)\,e^{-ct}}{(a-b)(b-c)(c-a)}. \qquad (24)$$

Application of equation (24) to equation (22) will result in the following: 1) the contribution of the poles at $z = -(u+\tau)$ and $z = -v$ will cancel out the first two terms of equation (20) (which were omitted from $H(x)$). 2) After fairly lengthy algebraic condensation one gets:

$$\frac{I(x)}{I(o)} = \int_0^x e^{-\tau x} F(x',x)\, dx' = \left[\frac{\sqrt{(u+v+\tau)^2 - 4v\tau} + (u+v) + \left(\frac{u-v}{u+v}\right)\tau}{2\sqrt{(u+v+\tau)^2 - 4v\tau}} \times \exp\left\{-\frac{x}{2}\left(u+v+\tau - \sqrt{(u+v+\tau)^2 - 4v\tau}\right)\right\}\right] \qquad (25)$$

$$+ \left[\frac{\sqrt{(u+v+\tau)^2 - 4v\tau} - (u+v) - \left(\frac{u-v}{u+v}\right)\tau}{2\sqrt{(u+v+\tau)^2 - 4v\tau}} \times \exp\left\{-\frac{x}{2}\left(u+v+\tau + \sqrt{(u+v+\tau)^2 - 4v\tau}\right)\right\}\right].$$

A study of equation 25 reveals the following:

a) $$\lim_{\tau\to 0}\left\{\frac{I(x)}{I(o)}\right\} = \int_0^x F(x',x)\, dx' = 1 \qquad (26)$$

fulfilling the expectation of equation 5;

b) Letting $u = \eta\mu$; $v = \eta\nu$, then

$$\lim_{\eta \to \infty} \frac{I(x)}{I(o)} = e^{-\frac{\nu_r}{\mu+\nu} x} \qquad (27)$$

that is, as the average solid and void segment goes to zero the reduction in intensity in the pebble stack becomes that of the homogeneous medium of reduced density;

c) $$\lim_{\tau \gg \eta} \frac{I(x)}{I(o)} = \frac{u}{u+v} e^{-vx} , \qquad (28)$$

this latter being the case for pebbles being totally opaque, and gives the probability of void arrays extending through the whole stack.

d) the second term decays much more rapidly than the first so that for any reasonable value of x it is essentially zero and in addition the coefficient of the exponential of the second term is small;

e) the coefficient of the first term is $\approx$ unity. For $u \approx v \gg \tau$, for instance, if $u = v = \tau$, the coefficient of the first term is 0.95, of the second .05; if $u = v = 4\tau$ the coefficient of the first term is .997, of the second .003. Therefore for $\tau \leq u \approx v$,

$$\frac{I(x)}{I(o)} = \exp\left\{ - \frac{x}{2}\left(u+v+\tau - \sqrt{(u+v+\tau)^2 - 4 v \tau} \right)\right\}. \qquad (29)$$

For $\tau > u \approx v$ the more general formula must be used.

One may also determine the average amount of solid material encountered in traversing the pebble bed at distance x:

$$\overline{x'} = \int_0^x x' \; F(x', x) \, dx' \qquad (30)$$

by the Laplace transform method. The integral again may be expressed in convolution form. The evaluation of the residues is somewhat more complicated since all of the poles of the inverse transformation are of the second order. There results (as indeed it must - from elementary considerations)

$$\overline{x'} = \frac{v}{u + v} \; x. \qquad (31)$$

That is, the average amount of solid material encountered is that amount expected from the homogeneous medium of reduced density.

4. Comparison of Theory with Experiment

One experiment designed to measure the channelling effect has been conducted by Palladino and Clifford[7] of the Oak Ridge National Laboratory. In their experiment the source of radiation

[7] O.R.N.L. Memorandum File No. 47-11-33, N. J. Palladino and C. E. Clifford, "Gamma Ray Attenuation Tests on Steel Plates and 1" Diameter Steel Punchings", 31 October, 1947.

for the experiment was the graphite moderated pile at ORNL. Part of a plug in a large hole roughly two feet square in cross-section through the shielding was removed to increase the radiation. In front of this hole was placed a specially constructed box in which could be inserted various thicknesses of iron plates or various thicknesses of iron punchings, one inch in diameter and for the most part a half inch in thickness (although there were appreciable numbers with thickness from 3/8" to 3/4"). The punchings were retained at front and back by a 1/4" iron plate. After correcting for the effect of the retaining plates the data are plotted as the points in Figure 3. For the case of the iron plates the actual distance was doubled to allow for an effective reduction in density of one-half -- corresponding to the observed ratio of average density of the punchings to that of the solid iron. A graph of $e^{-\tau x/2}$ is drawn with $\tau = 0.238\ cm^{-1}$ chosen to fit the data for the iron plates (solid line).

A theoretical estimation of the attenuation in the punchings requires an evaluation of the parameters u and v.

For the case of steel punchings 1 inch in diameter and 1/2 inch thick, we may estimate u from the Gauss theorem[8] which

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[8]It should be pointed out that the Gauss formula for the average chord length is correct only for the case of parallel beams of radiation - or in general when the radiation comes from sources at infinity. The so-called case of "isotropic radiation" refers to a radiation flux from sources at all solid angles at infinity.

For the case of pile radiation, where the sources are extended in space, sometimes all the way up to the surface of the volume under consideration, the Gauss formula does not give the true average chord length. This latter is a function of the pile geometry, the position in the pile, and the pile absorption, and is in general less than the average chord length $4V/S$ given by Gauss.

As an example consider the sphere. The average chord corresponding to the uniform source at ∞ is

$$\bar{l}_\infty = \int_0^{2r} l \frac{l}{2r^2} dl = \frac{4}{3} r.$$

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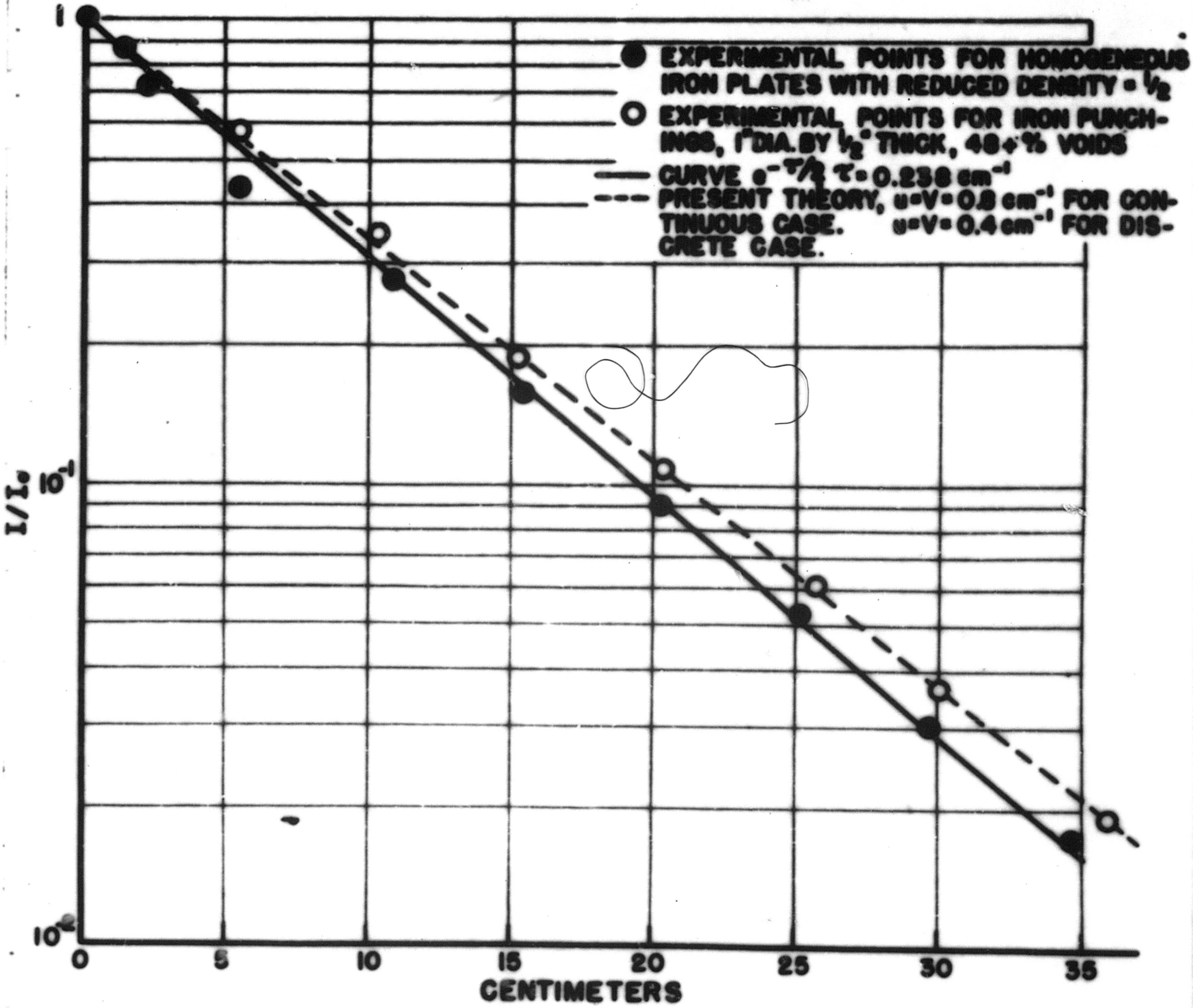


-FIG.3-

COMPARISON OF PRESENT THEORY AND EXPERIMENT BY CLIFFORD & PALLADINO ON PILE γ-RADIATION ATTENUATION IN IRON PUNCHINGS AND HOMOGENEOUS IRON PLATE.

For the case where the sources are on the surface of the sphere, the average chord length $\bar{l}_{contact}$ is given by

$$\bar{l}_{contact} = \frac{1}{2\pi} \int_0^{\pi/2} \rho \left[2\pi\rho \sin\theta\right]\left[\frac{\rho d\theta}{\rho^2}\right].$$

For a sphere $\rho = 2r \cos\theta$,

$$\bar{l}_{contact} = \int_0^{2r} l \left[\frac{dl}{2r}\right]$$

$$= r.$$

Therefore, when we apply the theory of finite voids to the case of diffusion, an average chord length somewhat less than $\bar{l}$ should be used. This discrepancy in $\bar{l}$ is of magnitude 10%. That is, we tend to overestimate the channelling effect for diffusion if Gauss theorem is applied. As this is of the nature of a correction to a correction, we shall continue to use Gauss' theorem for the average chord with the mental reservation that we can correct this value when the correction is warranted.

---

says the average chord in a solid (with no concavity in its surface) is given $4V/S$, where V is the volume and S the surface. For a right circular cylinder of radius r and height r, this becomes:

$$\frac{4\pi r^3}{2\pi r^2 + 2\pi r^2} = r.$$

Thus, if we set $u = 1/r$, we get

$$u = 0.8 \text{ cm}^{-1}.$$

From the experimental observation [7] that the fraction of voided volume is 48% + we may set $v = u = 0.8 \text{ cm}^{-1}$.

Figure 4 contains a graph of $F(x', 30 \text{ cm})$ for the case $u = v = 0.8 \text{ cm}^{-1}$, using equation 14.

The dotted line in Figure 3 has been computed using the above numerical values and equation 29, yielding $I(x)/I(o) = e^{-0.110x}$.

The experimental points and the theoretical curves agree reasonably well.

Since only this one experiment has been carried out one hesitates to make strong claims for the theory. At the very worst, however, the parameters u and v may be adjusted to fit one experiment and should give reasonable good fits for other values of $\lambda\tau$ (different size pebbles of the same shape or different radiation). Thus, one must carry out at least one experiment for every shape of pebble and u and v may be determined as proportional to some measurable dimension of the pebble.

The theory could be tested with monochromatic X-radiation (very small pebbles could then be used as τ would be high). Since the reduced density $\rho = v/(u+v)$ is very simple to measure one needs in principle only one attenuation experiment to compute u and v.

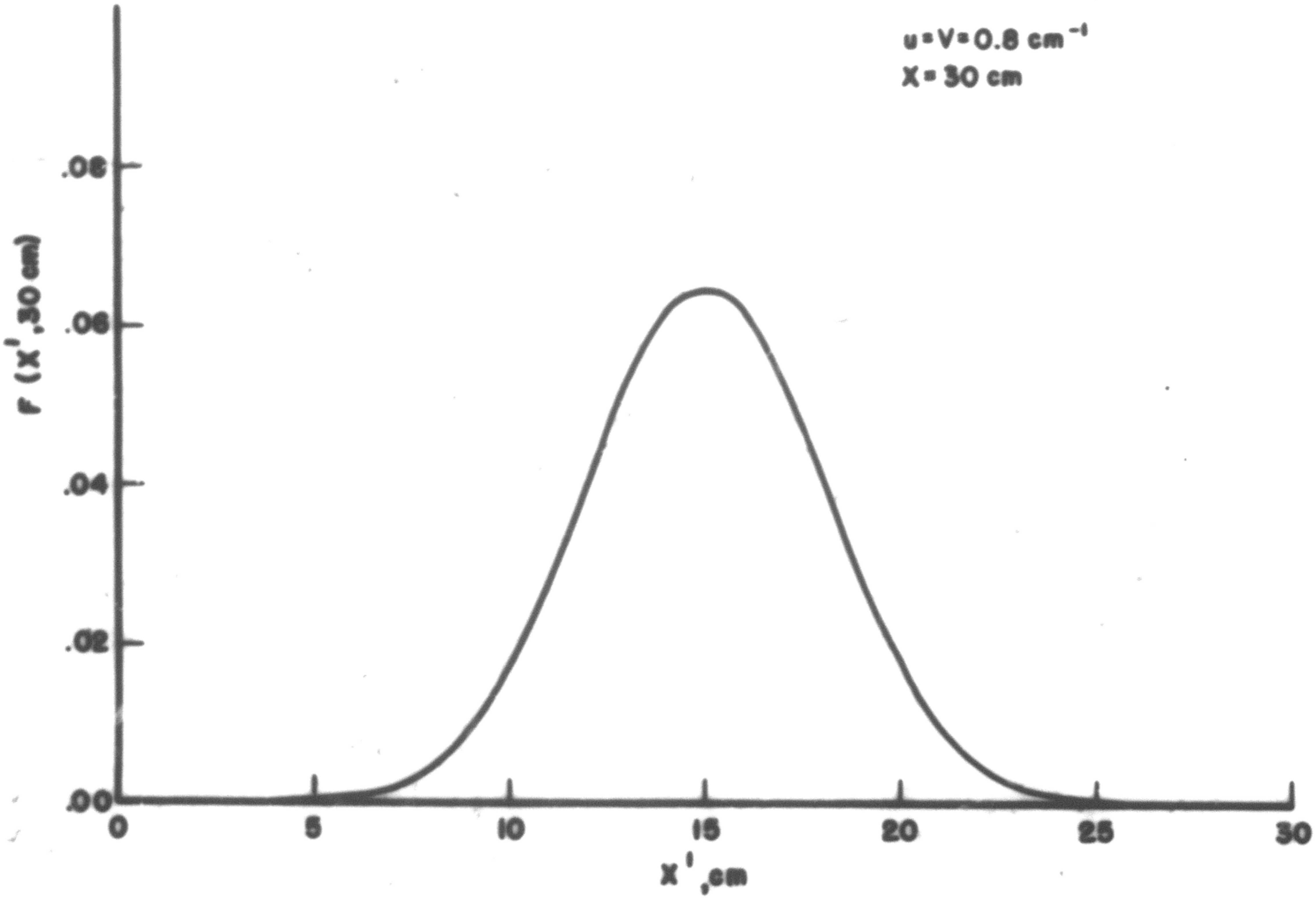
If $S(1,x)$ is taken to be a delta function:

$$S(1,x) = \delta(x-a),$$

then it is easy to show that

$$S(n,x) = \delta(x-na)$$

Drawing # 5276



-FIG. 4-

THE PROBABILITY OF PENETRATING X' cm OF SOLID MATERIAL IN TRAVERSING 30 cm OF PEBBLE BED HAVING $u = V = 0.8\ cm^{-1}$

and similarly

$$\psi(n, x) = \delta(x-nb) .$$

For a shield, laminated with material and void spaces -- as for instance spaced homogeneous plates -- the summation of equation 10 reduces to the one term for which $n = x/(a + b)$ and the resulting reduction in intensity given by

$$I(x)/I(o) = e^{-\tau na} = e^{-\tau ax/(a+b)},$$

that is, the reduction in intensity is that corresponding to the homogeneous material of reduced density.

One can, however, relax the constraint that the solid and void discrete unit-segments must alternate, and consider instead the probability of having a run of solid or void unit-segments. This is the nature of the development of this next section. Thus one builds up a run of alternate void lengths composed each of runs of unit-segments. The result is a theory involving discrete distributions. If in addition, the assumption of uniform probability is made, one expects results, on the whole, similar to that of the preceding section.

5. Alternate Model - Coveyou Formula

R. R. Coveyou has suggested an alternate model which shows the same behavior as equation 29.

Consider that all the pebbles are cubes and rigidly oriented in space. Consider further a cubic cell of size equal to the cubic pebble, and let ρ = probability that the cell is occupied by a cube. This is to be interpreted as the fraction of the total volume occupied by solid

material which we shall define as "reduced density". If λ = the size of a cell or cube, there will be z/λ cells in a line ray penetrating the shield from front to back.

The probability that k' cells are occupied is then

$$P(k') = \binom{z/\lambda}{k'} \rho^{k'} (1-\rho)^{z/\lambda - k'} .$$

The $\sum_{k'} P(k')$ is obviously unity.

Now the attenuation in each pebble is $e^{-\tau\lambda}$, where τ is the absorption coefficient, hence

$$I/I_o = \sum_{k'=0}^{z/\lambda} \binom{z/\lambda}{k'} \rho^{k'} \cdot (1-\rho)^{(z/\lambda)-k'} e^{-k'\tau\lambda}$$

$$= (1 - \rho + \rho e^{-\tau\lambda})^{z/\lambda} \qquad (32)$$

or

$$I/I_o = \exp\{ z/\lambda \ \ln \ [1 - \rho + \rho e^{-\tau\lambda}] \}. \qquad (32a)$$

Note that the exponent is negative. When the pebble size shrinks, it is easily shown that

$$\lim_{\lambda \to 0} (I/I_o) = e^{-z\rho\tau} \qquad (33)$$

Furthermore, the exponent of equation 32a is always less in absolute value than that of equation 33. That is, the attenuation in the pebbles is always less than that of the homogeneous case, and is exponential. This is in qualitative agreement with the previous theoretical results and substantiates our interpretation of the experimental results.

The Coveyou formula does not consider variation of solid segment thickness - a distribution which must augment the channelling effect. For pebble not of cubic shape and one where all orientations are possible, it is difficult to interpret what is meant by the average size λ. In addition the interpretation of λ is difficult when $\rho \neq \frac{1}{2}$. As far as the absorption is concerned the "average size" should be something less than the average chord ($4V/S$); but for the number of pebble spaces z/λ, and for $\rho = \frac{1}{2}$, the size of the pebble should be taken as some mean of the chords with the longer chords weighted more heavily. If in the absence of criteria, the mean chord $4V/S$ is used for λ, the errors will tend to compensate. Using, therefore, $\lambda = 1.25$ cm, $\rho = \frac{1}{2}$, $\tau = 0.238$ cm^{-1}, we got that

$$I/I_0 = e^{-.110z}$$

a line which is indistinguishable from the results of computations using equation 29.

Let us now consider the general theory in the light of the Coveyou model in order to get a more nearly complete understanding of the problem.

6. The Case of Uniform Probability with Segments of Discrete Length

When the distribution of the unit-chord lengths is a delta function, we must build up the primary distributions artificially. In so doing, a distribution is obtained which is not continuous, but which has values only on a discrete and uniform set of points. That is, one is constrained to take an integral number of unit-chords. In an analogous manner to the definitions given in Section 2 let

S(n,k) = the probability that n chords from the S(1,k) distribution sum up to be exactly k unit-chords in length, and

S(1,k) = the normalized probability that k solid unit-chords follow in sequence in random selection from the solid and void unit-chords.

Similar definitions are made for the V(n,k) distributions.

In any sequence of solid and void unit-chords, there will be in general a run of solid, then a run of void unit-chords. Taking each run as a whole gives then the S(1,k) and V(1,k) distributions. The problem is of exactly the same nature as for the case of continuous distribution except that the integrals are now replaced by summations. In particular, we may also consider a case of uniform probability as we did to obtain the exponential distribution of equation 7.

Consider a chord made up of several unit-chords, and let the probability that a given unit-chord in the last segment in this sequence be p. Then

$$S(1,k) = p(1-p)^{k-1}. \tag{34}$$

We know that we have at least one unit-chord. The probability that it is followed by k - 1 unit chords is $(1-p)^{k-1}$ and p is the probability that the last one terminates the sequence. It might appear that S(1,k) should be given by $(1-p)^k$, however when S(1,k) is normalized the above equation results.

Consider a space k units long, divided into two parts, those units from 0 to k', and from k' to k. The probability that one S(1,k) chord k' units long fills the interval from 0 - k' and that (n - 1)S(1,k) chords fill the interval from k' to k is the product

$$S(n-1,\ k-k')\ S(1,k')$$

and, therefore,

$$S(n,k) = \sum_{k'=1}^{k-n+1} S(n-1,\ k-k')\ S(1,k'). \tag{35}$$

The upper limit of the summation results from the fact that we must allow at the upper limit space for n - 1 of the S(1,k) chords, each of at least one unit-chord in length.

Using now the S(1,k) resulting from the assumption of uniform probability, we get

$$S(n,k) = \binom{k-1}{n-1} p^n (1-p)^{k-n} . \qquad (36)$$

Equation 36 may be proved by induction, but inasmuch as the induction proof itself involves the use of generating functions we shall employ the generating function directly which allows an evaluation in a manner which is entirely operational; i.e. without recourse to induction and the intuition required to guess the correct solution.

We first point out the analogy between the so-called generating functions and the Laplace Transform. The generating function for the S(1,k) distribution is defined

$$s(1,x) = \sum_{k=1}^{\infty} x^k S(1,k). \qquad (37)$$

We shall henceforth refer to equation 37 as a transformation, and the function s(1,x) as the transform of S(1,k). The inverse transformation is very simply obtained:

$$S(1,k) = ((x^k)) \quad s(1,x) \qquad (38)$$

where $((x^k))$ means "the coefficients of x^k in the following, i.e. in the expansion of s(1,x) as a power series in x.

Note the analogy between these transformations and the corresponding operations for the Laplace transformation. The variable x occupies a similar position as e^{-s}. The inverse operation may be written

$$S(1,k) = ((x^{-1})) \; [\, x^{-k-1} \; s(1,x)], \qquad (38a)$$

an operation which has similarity to the evaluation of the complex integral of an inverse Laplace transformation.

Letting the symbol G{S(1,k)} represent the transformation of 37 and G^{-1} (s(1,x)) the inverse transformation we note that an analogous convolution theorem holds for the generating transformation:

$$G \left\{ \sum_{k'=1}^{k-n+1} S(n-1,\, k-k')\, S(1,k') \right\} = s(n-1,x)\; s(1,x), \qquad (39)$$

since

$$G \left\{ \sum_{k'=1}^{k-n+1} S(n-1,\, k-k')\, S(1,k') \right\}$$

$$= \sum_{k=1}^{\infty} x^k \sum_{k'=1}^{k-n+1} S(n-1,\, k-k')\, S(1,k')$$

$$= \sum_{k=1}^{\infty} \sum_{k'=1}^{k-n+1} x^{k-k'}\, S(n-1,\, k-k')\, x^{k'}\, S(1,k')$$

and, by reversing the order of the summation

$$= \sum_{k'=1}^{\infty} \sum_{k=k'+1}^{\infty} x^{k-k'}\, S(n-1,\, k-k')\, x^{k'}\, S(1,k')$$

$$= \sum_{k-k'=1}^{\infty} x^{k-k'}\, S(n-1,k-k') \sum_{k'=1}^{\infty} x^{k'}\, S(1,k')$$

$$= s(n-1,\, x)\quad s(1,x).$$

Now, by means of the recursion relation (equation 35), we have

$$s(n,x) = [s(1,x)]^n. \qquad (40)$$

Using the expression for s(1,x) (equation 34) one gets:

$$s(1,x) = px\,[1 + (1-p)x + (1-p)^2\,x^2 + \ldots\ldots]$$

$$= \frac{px}{1-(1-p)x}.$$

And, therefore

$$s(n,x) = \frac{(px)^n}{[1-(1-p)\,x]^n} \qquad (40a)$$

and the inverse transformation

$$p^{-n}\,S(n,k) = ((x^k))\;\frac{x^n}{(1-(1-p)x)^n}$$

$$= ((x^{k-n}))\,[1-(1-p)\,x]^{-n},$$

$$S(n,k) = \binom{k-1}{n-1}\,p^n\,(1-p)^{k-n}. \qquad (3c)$$

Similarly, one gets

$$V(n,k) = \binom{k-1}{n-1}\,q^n\,(1-q)^{k-n}. \qquad (41)$$

Note that S(n,k), being normalized, always contains the implication that the segment represented by this function is followed by a voided segment. When we get to the end of the interval we do not in general care whether the next segment begins with a solid or void.

Now the distribution of solid and void segments which result from random drawings from a mixture of solid and void unit segments is simulated by two drawings - one from S(n,k) and one from V(n,k). All of the solid (S(1,k) segments have been pushed together from 0 to k', and all of the void V(1,k) segments are assembled in the interval k' to k.

The "experimental" analogy to the mathematical formulation is as follows. We substitute the word "black" for "solid" and "white" for "void". We have a large (infinite) bin full of black and white -- and grey -- sticks all of the same length and fabricated in such a manner that they can be made to stick together like tinker toys. The grey linkages have the remarkable property that they turn black when attached to a black stick; white if attached to a white stick. A fraction q of the sticks are black; a fraction p are white; all of the remainder are grey. Note that p + q in general is less than unity.

We call an assistant and keep him busy for several days at drawings from the bin. He starts by drawing until he gets a black or white stick. Suppose it is black. If the next drawing is grey or black he attaches it onto the first, etc., thus building up a length of k black sticks (since the grey ones have turned black). Whenever he draws a white stick, he throws the attached linkages into the S(1,k) bin and proceeds in a similar way with his white sticks, filling the V(1,k) bin with them.

We then take a second assistant who makes two drawings from the S(1,k) bin, fastens them together and throws the combined linkage in the S(2,k) bin. A drawing from the S(1,k) and S(n-1,k) bin -- or the S(m,k) and S(n-m,k) bin --

are fastened together and put in the S(n,k) bin.

A similar process is carried out for the V(n,k) bins.

Now the foreman comes down and makes two drawings, one from the S(n,k) bin and one from the V(n,k) bin, or V(n-1,k or V(n+1,k) bin. In the diagram, Figure 5, one set of drawings is made for each line which links jars of segments of opposite color.

The probability that alternate selections of void and solid S(1,k) and V(1,k) segments add up to exactly k is replaced by the probability of drawing just once from the S(n,k) and from the V(n+1,k); V(n,k) or V(n-1,k) bin. Call this probability H(k).

$$H(k) = \sum_{k'} F(k',k).$$

$$\begin{aligned} F(k',k) &= \frac{q}{p(p+q)} S(1,k) \delta_{k',k} \\ &+ \frac{q}{p(p+q)} \sum_{n} S(n+1,k') V(n,k-k') \\ &+ \frac{2}{p+q} \sum_{n} S(n,k') V(n,k-k') \qquad (42) \\ &+ \frac{p}{q(p+q)} \sum_{n} S(n,k') V(n+1,k-k') \\ &+ \frac{p}{q(p+q)} V(1,k) \delta_{k',o}. \end{aligned}$$

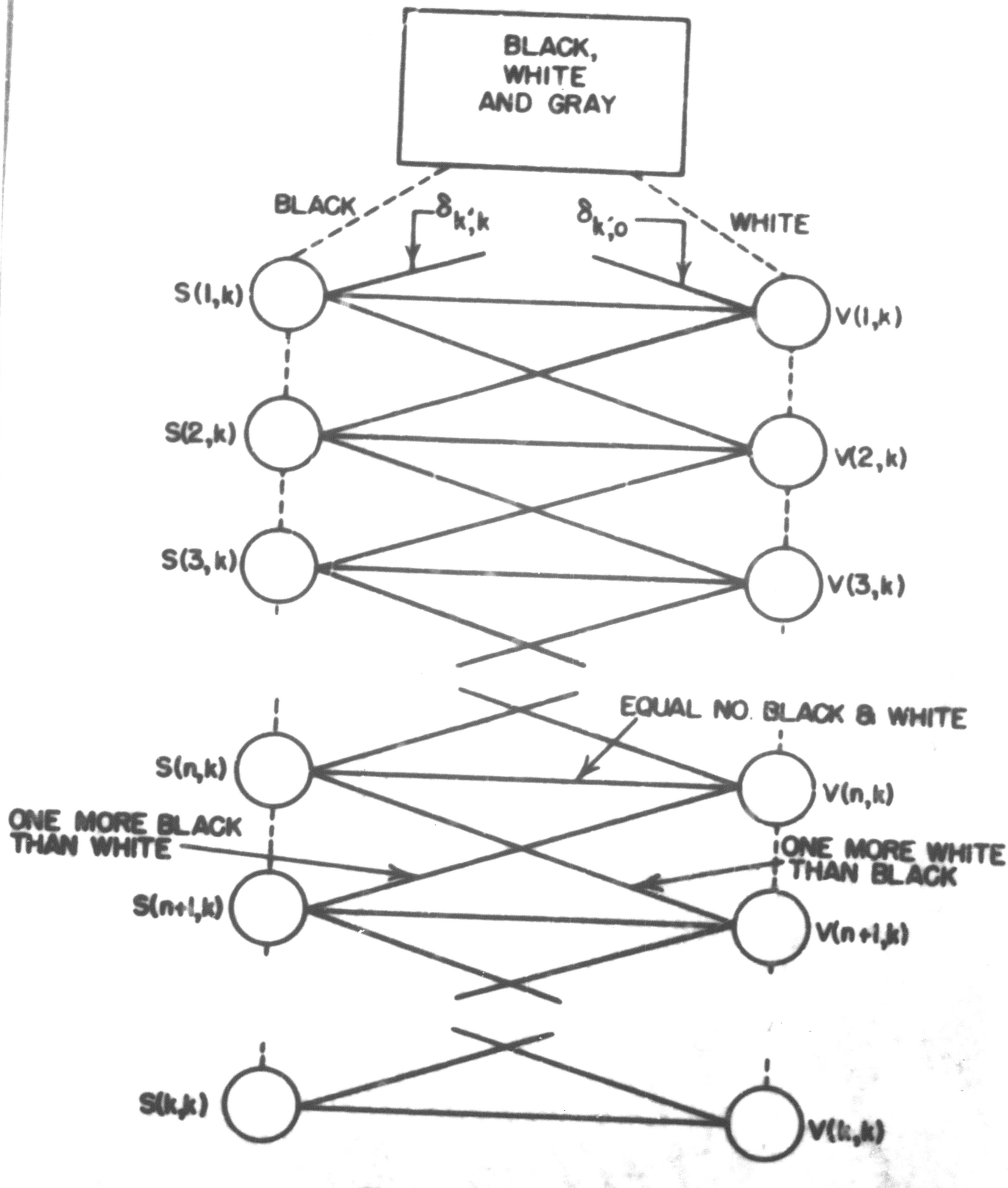


Fig. 5

Illustration of the probability sums as an urn problem. Dotted line indicates drawing for filling of urns. Solid lines indicate combination drawing from two urns.

The terms including the Kronecker delta are for cases where $k' = k$, and $k' = 0$ when the ray contains all solid or all void material respectively.

In the first double summation of $H(k',k)$ we have one less void segment (one less $V(1,k)$ that is) than solid segment. That is, we start and end with an $S(1,k)$ segment. Therefore, the term is multiplied by the probability of starting with a solid segment $\frac{q}{p+q}$. Inasmuch as these terms represent ending with a solid segment we note that the last one contains implicitly the probability p that it is followed by a void segment. As we do not care whether it is followed by void or solid as we quit when k units are filled, we remove this factor by dividing by p. As there are alternate solid and k void $S(1,k)$ and $V(1,k)$ segments inside the interval, their terms cancel out as they are always followed by a segment of opposite type.

The second summation contains all those terms which begin with one type and end with another. The terms beginning with a solid segment must be multiplied by $\frac{q}{p+q}$ for the probability of starting with a solid, and divided by q to remove the necessity of the last segment being followed by a solid. Similarly the terms starting with a void and ending with a solid are multiplied by $\frac{p}{p(p+q)}$. The sum is $2/(p+q)$.

Analogously the last summation is multiplied by $\frac{p}{q(p+q)}$ since it represents starting with a void and ending with a void.

The summations are carried out over all the range for which the argument has values.

For convenience in notation let $p+q = t$ and $1-t=r$. We define a function $G(k)$, and substituting the values of $S(1,k)$ and $V(1,k)$

$$H(k) = \frac{q}{t}(1-p)^{k-1} + \frac{p}{t}(1-q)^{k-1} + G(k). \qquad (43)$$

That is G(k) contains those terms involved in the double summations. Making the generating function transformation $g(x)$ of $G(k)$ we get (by convolution theorem, equation 39)

$$g(x) = \frac{q}{pt}\sum_{n=1}^{\infty} s(n+1,x)\ v(n,x)$$

$$+ \frac{2}{t}\sum_{n=1}^{\infty} s(n,x)\ v(n+1,x)$$

$$+ \frac{p}{qt}\sum_{n=1}^{\infty} s(n,x)\ v(n+1,x). \qquad (44)$$

From Equation 40, the transform of S(n,k) -- and similarly for the transform of V(n,k) we get on substituting into equation 44, and further noting that the transform of the first two terms in 43 may be included in the first and third summations:

$$h(x) = \frac{q}{pt}\sum_{n=0}^{\infty}\left[\frac{px}{1-(1-p)x}\right]^{n+1}\left[\frac{qx}{1-(1-q)x}\right]^{n}$$

$$+ \frac{2}{t}\sum_{n=1}^{\infty}\left[\frac{px}{1-(1-p)x}\right]^{n}\left[\frac{qx}{1-(1-q)x}\right]^{n}$$

$$+ \frac{p}{qt}\sum_{n=0}^{\infty}\left[\frac{px}{1-(1-p)x}\right]^{n}\left[\frac{qx}{1-(1-q)x}\right]^{n+1}. \qquad (45)$$

On performing the indicated summations and simplifying, one gets, since

$$\sum_{n=0}^{\infty} z^n = (1 - z)^{-1} \quad \text{and} \quad \sum_{n=1}^{\infty} z^n = z(1 - z)^{-1} ;$$

$$h(x) = \frac{qx[1 - (1-q)x]}{t\,(1-x)\,(1-rx)} + \frac{2pq\,x^2}{t(1-x)\,(1-rx)} + \frac{p\,x[1 - (1-p)x]}{t(1-x)\,(1-rx)} \quad . \qquad (46)$$

One needs the power series expansion of $(1-x)^{-1}\,(1-rx)^{-1}$:

$$\frac{1}{1-x} \cdot \frac{1}{1-rx} = \sum_{m=0}^{\infty} x^m \cdot \sum_{l=0}^{\infty} (rx)^l$$

$$= \sum_{m=0}^{\infty} x^m \sum_{j=0}^{m} r^j$$

$$\text{but} \sum_{j=0}^{m} r^j = \sum_{j=0}^{\infty} r^j - \sum_{j=m+1}^{\infty} r^j$$

$$= \frac{1-r^{m+1}}{1 - r} \quad . \qquad (47)$$

and is therefore the coefficient of x^m.

Making the inverse transformation (the coefficient of x^k) one gets:

$$H(k) = \frac{q}{t}\, r^{k-1} + \frac{q^k}{t^2}\,(1 - r^{k-1})$$

$$+ \frac{2pq}{t^2}\,(1-r^{k-1})$$

$$+ \frac{p}{t}\, r^{k-1} + \frac{p^2}{t^2}\,(1-r^{k-1})$$

$$H(k) = 1. \qquad (48)$$

Thus $F(k', k)$ is normalized for all values of the parameter $t=p+q$.

To compute the transmitted intensity we must evaluate the summation, which for unit incident intensity is:

$$I(k) = \sum_{k'} e^{-\beta k'} \quad F(k', k), \qquad (49)$$

where β is the linear absorption coefficient τ multiplied by the length λ of the unit segment. Equation 49 may be written in convolution form if we group the exponential with the $S(n,k)$ distributi- We define

$$W(n,k) = e^{-\beta k} \quad S(n, k)$$

$$= \binom{k-1}{n-1} (pe^{-\beta})^n \,[e^{-\beta}(1-p)]^{k-n} \quad . \qquad (50a)$$

The transform $w(n,x)$ of $W(n,k)$ is therefore

$$w(n,x) = \left[\frac{e^{-\beta}\,px}{1-e^{-\beta}\,(1-p)\,x}\right] . \qquad (50b)$$

The transform i(x) of I(k) becomes:

$$i(x) = \frac{q}{pt}\sum_{n=0}^{\infty}\left[\frac{e^{-\beta}\,px}{1-e^{-\beta}\,(1-p)x}\right]^{n+1}\left[\frac{qx}{1-(1-q)\,x}\right]^{n}$$

$$+\ \frac{2}{t}\sum_{n=1}^{\infty}\left[\frac{e^{-\beta}\,px}{1-e^{-\beta}\,(1-p)x}\right]^{n}\left[\frac{qx}{1-(1-q)x}\right]^{n}$$

$$+\ \frac{p}{qt}\sum_{n=0}^{\infty}\left[\frac{e^{-\beta}\,px}{1-e^{-\beta}\,(1-p)x}\right]^{n}\left[\frac{qx}{1-(1-q)x}\right]^{n+1}. \quad (51)$$

The transforms of the two terms not under summation signs have been absorbed in the first and last term above by setting the lower limit of n at zero.

On performing the indicated summation we get:

$$i(x) = \frac{q\,e^{-\beta}x\,[1-(1-q)x]}{t(1-ax)(1-bx)} + \frac{2\,e^{-\beta}\,pq\,x^2}{t(1-ax)(1-bx)} + \frac{p\,x[1-e^{-\beta}(1-p)\,x]}{t(1-ax)(1-bx)}, \quad (52)$$

where a and b are the reciprocals of the roots of the equation

$$1 - [e^{-\beta}(1-p) + (1-q)]\,x + e^{-\beta}(1-t)\,x^2 = 0. \quad (53a)$$

Solving

$$x_1 = \frac{e^{-\beta}(1-p)(1-q) + \sqrt{[e^{-\beta}(1-p) + (1-q)]^2 - 4\,e^{-\beta}(1-t)}}{2\,e^{-\beta}(1-t)}; \quad (53b)$$

$$x_2 = \frac{e^{-\beta}(1-p) + (1-q) - \sqrt{[e^{-\beta}(1-p) + (1-q)]^2 - 4\,e^{-\beta}(1-t)}}{2\,e^{-\beta}(1-t)}; \quad (53c)$$

and

$$a = \frac{1}{x_1}; \quad b = \frac{1}{x_2}. \quad (53$$

We note that

$$a + b = e^{-\beta}(1-p) + (1-q),$$

$$b - a = \sqrt{[e^{-\beta}(1-p) + (1-q)]^2 - 4\,e^{-\beta}(1-t)}, \quad (54)$$

$$ab = e^{-\beta}(1-t).$$

To perform the inverse transformation we need the expansion of

$$\frac{1}{1-ax} \cdot \frac{1}{1-bx}$$

$$= \sum_m^\infty (ax)^m \cdot \sum_j^\infty (bx)^j$$

$$= \sum_m x^m \sum_{j=0}^{m} a^{m-j}\, b^j$$

$$= \sum_m x^m \frac{a^{m+1} - b^{m+1}}{a - b} \qquad (55)$$

Using equation 55 to determine the coefficient of x^m we carry out the inverse transformation, which on simplifying becomes

$$I(k) = \frac{a^k}{a-b}\left\{\frac{q\,e^{-\beta} + p}{t} + \frac{e^{-\beta}(t-1)}{a}\right\}$$

$$- \frac{b^k}{a-b}\left\{\frac{q\,e^{-\beta} + p}{t} + \frac{e^{-\beta}(t-1)}{b}\right\}. \qquad (56)$$

For the case where there is no dilution of the "white" and "black" unit sticks; i.e. where $p+q = t = 1$, we note from equation 53 that $a = 0$ and $b = e^{-\beta}(1-p) + 1 - q = e^{-\beta} q + p$, and therefore

$$\underset{t\to 1}{I(k)} = (q\,e^{-\beta} + p)^k. \qquad (57)$$

A result which is identical with the Coveyou formula (equation 32) since $q = \rho$; $p = \rho$ and $\beta = \tau\lambda$ where λ is the length of a unit segment.

7. The Continuous Case as a Limit of the Discrete Case.

We desire now to allow t to go to zero in such a manner that the average solid and void segments in the $S(1,k)$ and $V(1,k)$ distribution remain constant. Letting λ be the length in centimeters of the unit segment then these average values are

$$\bar{S} = \frac{\lambda}{p}$$

$$\bar{V} = \frac{\lambda}{q}$$

Therefore if we set

$$p = u\lambda$$

$$q = v\lambda$$

$$\beta = \tau\lambda$$

$$\tau = (u+v)\lambda$$

and

$$k = z/\lambda$$

and take the limit $\lambda \to 0$ we expect that the discrete case will go to the continuous case in the limiting process.

Keeping in mind that we shall allow λ to go to zero, we make the approximation

$$e^{-\tau\lambda} \doteq 1 - \tau\lambda + \frac{(\tau\lambda)^2}{2} \quad \text{- etc,}$$

$$a - b \doteq \lambda\,[(u + v + \tau)^2 - 4v\tau]^{\frac{1}{2}},$$

$$a \doteq 1 - \left[\frac{u + v + \tau}{2} - \frac{[(u + v + \tau)]^2 - 4v\tau]^{\frac{1}{2}}}{2}\right]\lambda,$$

$$b \doteq 1 - \left[\frac{u + v + \tau}{2}\right] + \left[\frac{[(u + v + \tau)^2 - 4v\tau]^{\frac{1}{2}}}{2}\right]\lambda$$

$$\frac{q\,e^{-\tau\lambda} + p}{\tau} \doteq 1 - \frac{\tau v}{u+v}\,\lambda,$$

$$\frac{e^{-\tau\lambda}\,(t - 1)}{a} \doteq -1 + \left[-\frac{u + v + \tau}{2} + \frac{[(u + v + \tau)^2 - 4v\tau]^{\frac{1}{2}}}{2}\right]\lambda,$$

$$\frac{e^{-\tau\lambda}\,(t - 1)}{b} \doteq -1 + \left[\frac{u + v + \tau}{2} - \frac{[(u + v + \tau)^2 - 4v\tau]^{\frac{1}{2}}}{2}\right]\lambda.$$

Combining

$$I(k) \doteq \frac{[(u+v+\tau)^2 - 4v\tau]^{\frac{1}{2}} + (u+v) + \left(\frac{u-v}{u+v}\right)\tau}{2[(u+v+\tau)^2 - 4v\tau]^{\frac{1}{2}}}\left\{1 - \left(u+v+\tau - \sqrt{(u+v+\tau)^2 - 4v\tau}\right)\frac{\lambda}{2}\right\}^{z/\lambda}$$

$$+ \frac{[(u+v+\tau)^2 - 4v\tau]^{\frac{1}{2}} - (u+v) - \left(\frac{u-v}{u+v}\right)\tau}{2[(u+v+\tau)^2 - 4v\tau]^{\frac{1}{2}}}\left\{1 - \left(u+v+\tau + \sqrt{(u+v+\tau)^2 - 4v\tau}\right)\frac{\lambda}{2}\right\}^{z/\lambda}.$$

Since

$$\lim_{\lambda \to 0}\,(1 - \gamma\lambda)^{z/\lambda} = e^{-\gamma z}$$

we have

$$\lim_{\lambda \to 0} I(k) = \frac{\sqrt{(u+v+\tau)^2 - 4v\tau} + (u+v) + \left(\frac{u-v}{u+v}\right)\tau}{2\sqrt{(u+v+\tau)^2 - 4v\tau}}\exp\left\{-\frac{z}{2}\left(u+v+\tau - \sqrt{(u+v+\tau)^2 - 4v\tau}\right)\right\}$$

$$+ \frac{\sqrt{(u+v+\tau)^2 - 4v\tau} - (u+v) - \left(\frac{u-v}{u+v}\right)\tau}{2\sqrt{(u+v+\tau)^2 - 4v\tau}}\exp\left\{-\frac{z}{2}\left(u+v+\tau + \sqrt{(u+v+\tau)^2 - 4v\tau}\right)\right\}.$$

That is, in the limit, the equation for the intensity of transmitted radiation goes from the discrete to the continuous case.

8. The Distribution of Solid Material.

The distribution of solid material for the case where $p+q = 1$ is of interest in the completion of the picture. Setting $q = \rho$; $p = 1 - \rho$, we have

$$F(k', k) = \rho^k \delta_{k,k'} + (1-\rho)^k \delta_{k',0}$$

$$+ \sum_n \left\{ \binom{k'-1}{n-1} \binom{k-k'-1}{n-2} + 2\binom{k'-1}{n-1}\binom{k-k'-1}{n-1} + \binom{k'-1}{n-1}\binom{k-k'-1}{n} \right\} \rho^k \quad (1-\rho)^{k-k'} \quad ,$$

$$= \rho^k \delta_{k,k'} + (1-\rho)^k \delta_{k',0}$$

$$+ \rho^{k'} (1-\rho)^{k-k'} \sum_n \sum_{\nu=0}^{2} \binom{k'-1}{n-1}\binom{k-k'-1}{n-\nu}\binom{2}{\nu} . \quad (60)$$

Now $\sum_{\nu=0}^{2} \binom{k-k'-1}{n-\nu}\binom{2}{\nu}$ is in convolution form. Its transform is, therefore

$$(1+x)^2 (1+x)^{k-k'-1} = (1+x)^{k-k'-1}$$

and the inverse transform - the coefficient of x^n - is

$$\sum_{\nu=0}^{2} \binom{k-k'-1}{n-\nu}\binom{2}{\nu} = \binom{k-k'+1}{n} . \quad (61)$$

The sum $\sum_n \binom{k'-1}{n-1}\binom{k-k'+1}{n}$

may be written in convolution form, since

$$\binom{k'-1}{n-1} = \binom{k'-1}{k'-n} , \text{ giving}$$

$$\sum_n \binom{k'-1}{k'-n}\binom{k-k'+1}{n} . \quad (62)$$

The transform of the sum (62) is then

$$(1+x)^{k'-1} \quad (1+x)^{k-k'+1} = (1+x)^k .$$

The inverse transform is the coefficient of $x^{k'}$

$$\sum_n \binom{k'-1}{k'-n}\binom{k-k'+1}{n} = \binom{k}{k'} . \quad (63)$$

Thus we have demonstrated that (for $p + q = 1$)

$$F(k,k') = \binom{k}{k'} \rho^{k'} \quad (1-\rho)^{k-k'} \quad (64)$$

(the Kronecker delta terms being absorbed).

To express the discrete case formula in terms of length rather than k we define

u^{-1} = ave. length of S(1,k) distribution

(65)

v^{-1} = ave. length of V(1,k) distribution.

Therefore

$$u^{-1} = \lambda p^{-1}$$

$$v^{-1} = \lambda q^{-1} \qquad (66)$$

since p^{-1} is the average number of unit segments in $S(1,k)$, etc.

From 66 and the fact that $p + q = 1$, we get

$$\lambda = (u + v)^{-1}$$

$$p = u(u + v)^{-1} \qquad (67)$$

$$q = v(u + v)^{-1}.$$

We therefore, replace λ by $(u + v)^{-1}$ to write equations 32 and 64,

$$\frac{I(z)}{I(o)} = \exp\left\{z(u + v)\ \ln\left[1 - \frac{v}{u+v} + \frac{v}{u+v}\,e^{-\frac{1}{u+v}}\right]\right\} \qquad (68a)$$

$$F(z',z) = \binom{z(u + v)}{z'(u + v)}\left(\frac{v}{u + v}\right)^{z'(u + v)}\left(\frac{u}{u + v}\right)^{(z-z')(u+v)}. \qquad (68b)$$

Equation (68b) has been compared with values obtained from the case of continuous distribution (Equation 12) for $u = 0.3\ cm^{-1}$ and $v = 0.6\ cm^{-1}$. For the same values of u and v the "discrete" case does not give the same distribution as the continuous case; but, instead, gives a distribution with less dispersion. If, however, $\frac{1}{2}(u + v)$ is used to replace $u + v$ [$\frac{1}{2}$ v for v] very good agreement is obtained. This is illustrated in Figure 6.

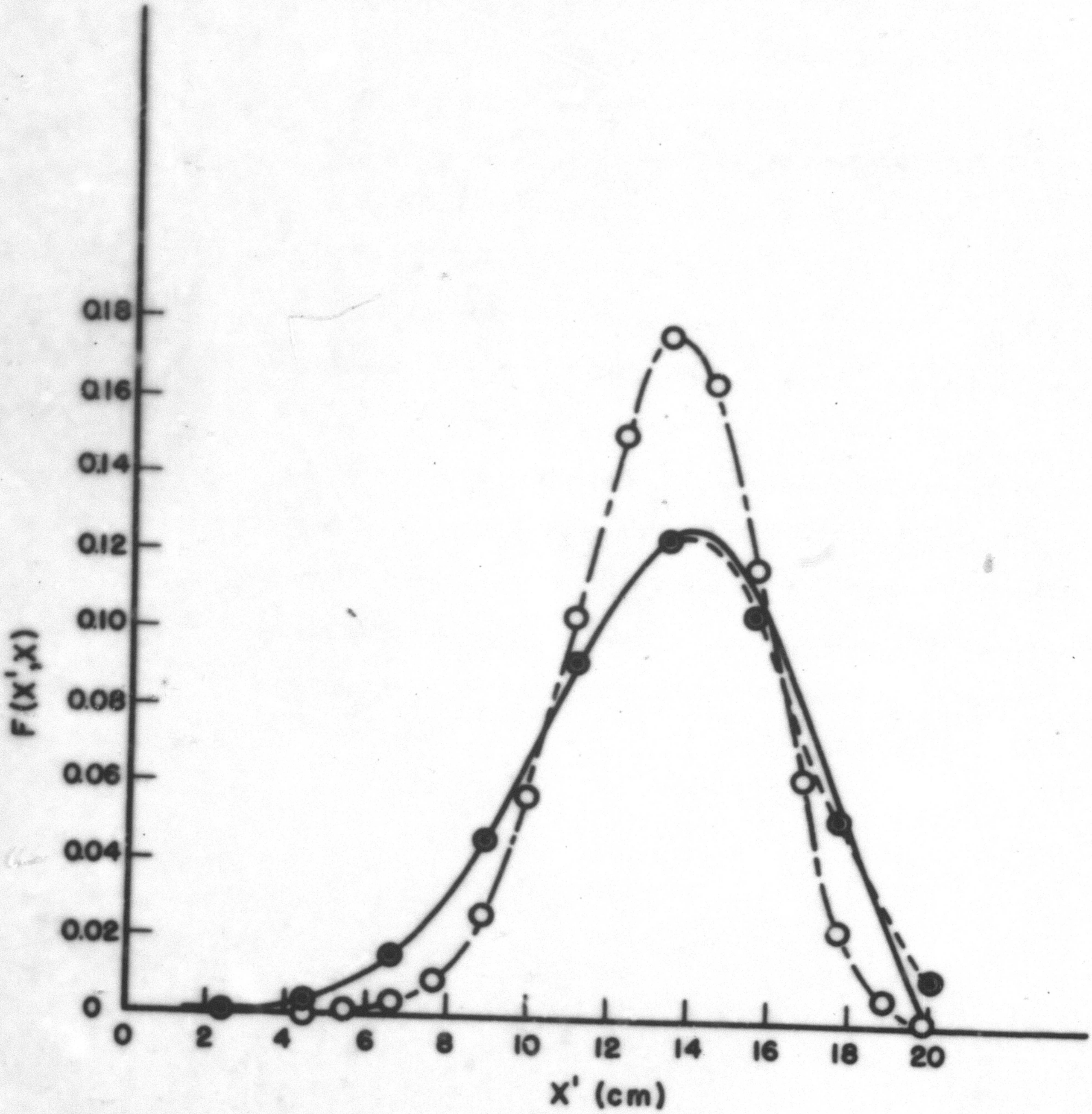
The curves suggest the cause for this effect. One can compute the standard deviation of the two distributions for comparison.

Figure 6

This comparison of the distributions of solid material penetrated in traversing 20cm. of a pebble bed as given by the continuous distribution and by the discrete distribution ($p + q = 1$) illustrates the difference in dispersion given by the two viewpoints. In the text it is shown that the standard deviation for the continuous case is approximately the square root of two times that for the discrete case for the same values of the parameters. Also illustrated is the fact that if the parameters for the discrete case are halved a good fit results; in this case particularly in the region of low x' -- the region of greatest importance. Were the ratio of u and v inverted, corresponding to a reduced density of 1/3 instead of 2/3, the curves would be mirror image of the above, since the equations are symmetric in u and v. However when the functions are weighted by the exponential in x', a greater discrepancy between the probability of penetration would result.

SECRET

Drawing # 5279



-FIG. 6-

COMPARISON OF CONTINUOUS DISTRIBUTION AND DISCRETE THEORIES

——— CONTINUOUS DISTRIBUTION; $u = 0.3 cm^{-1}$; $v = 0.6 cm^{-1}$

●- - -● DISCRETE DISTRIBUTIONS, $u = 0.15 cm^{-1}$; $v = 0.3 cm^{-1}$

○- - -○ DISCRETE DISTRIBUTIONS; $u = 0.3 cm^{-1}$; $v = 0.6 cm^{-1}$

The second moment of the continuous distribution is given by

$$\overline{(x')^2} = \int_0^x (x')^2 \; F(x,x')dx' \qquad (69a)$$

and the standard deviation, σ, by

$$\sigma^2 = \overline{(x')^2} - (\overline{x'})^2 \quad . \qquad (69b)$$

On substitution of $F(x',x)$ in 69a the result may be expressed in convolution form and the integral evaluated - using many pages of computation - by the Laplace Transform method. Using the value of $\overline{x'}$ already computed, Equation 30, there results

$$\sigma^2 = \frac{2uv}{(u+v)^4}\left(e^{-(u+v)x}-1\right) + \left(\frac{2uv}{(u+v)^3} + \frac{u^2v(v-2u)}{(u+v)^5}\, e^{-(u+v)x}\right)x. \qquad (70)$$

Or, when x is sufficiently large

$$\sigma^2 = \frac{2uv}{(u+v)^3}\, x. \qquad (71)$$

The standard deviation, Σ, for the discrete distribution is well known[9], since $F(k', k)$ is binomial.

[9] p.471 Fry, "Probability and Its Engineering Uses", D. Van Nostrand, 1928

$$\Sigma^2 = k\,p\,(1-p). \qquad (72)$$

To convert this into length rather than numbers of unit segments Equation 72 is multiplied by $(u + v)^{-2}$, that is by λ^2, k replaced by $x(u+v)$; p by $u/(u+v)$ and $(1-p)$ by $v/(u+v)$ getting

$$\lambda^2 \Sigma^2 = \frac{uv}{(u+v)^3} x. \qquad (73)$$

Thus, to get the same dispersion in both cases the u and v taken in the discrete case must be one-half the u and v in the continuous case.

Therefore, to get the same numerical result, when using the continuous distribution then one may estimate u_c^{-1} (the subscript c for continuous distribution, d for discrete distribution with $p+q=1$) by taking it to be the Gauss average chord $4V/S$; when using the discrete distribution u_d^{-1} is estimated by taking it to be twice u_c^{-1} or $8\,V/S$. This interpretation is not consistent with the interpretation given by the Coveyou formulation, where $\lambda = (u_d+v_d)^{-1}$ is taken to be the average pebble size or $4V/S$. This is summarized as follows: From the general theory one gets

$$u_c^{-1} = \frac{u_d^{-1}}{2} : \frac{4V}{S}$$

$$v_c^{-1} = \frac{v_d^{-1}}{2} : \frac{\rho}{1-\rho}\,\frac{4V}{S'} .$$

From the Coveyou formulation one gets

$$u_c^{-1} = (1-\rho)\, u_d^{-1} : \frac{4V}{S'},$$

$$v_c^{-1} = (1-\rho)\, v_d^{-1} : \frac{\rho}{1-\rho}\,\frac{4V}{S'}$$

There is, from the present discussion, no way to choose between these two interpretations, and one sees that the experimental test was fortuitous in that $\rho = \frac{1}{2}$, and the two interpretations give identical results.

Since distributions for the S and V functions are fictitious, and serve only to build a model for computation purposes there remains then the necessity of determining the parameters experimentally. One does not need to determine these for every size of pebble of a given shape, however, but for only one size, since the u^{-1} and v^{-1} will be linearly related to a measurable dimension, and the ρ will be constant.

Measurements could be taken with monochrometic x-rays for various shapes and sizes of pebbles so that the u^{-1} and v^{-1} values could be given for the different shapes for handbook computation.

9. Engineering Procedures.

In order to design a pebble shield of minimum weight and thickness or to design a pebble pile the channelling effect must be known. To determine this effect it is desirable to make one or more experimental measurements of attenuation in pebbles of the type chosen. It is pointed out that the models herein presented both predict that the factor by which the thickness is increased over the density effect (here defined as the channelling effect factor) is the same for pebbles having the same shape, and for which $\tau\lambda$ is the same -- where τ is the absorption or scattering coefficient and λ is a parameter proportional to a linear dimension of the pebble, i.e. is $(u+v)^{-1}$.

If we call f the channelling effect factor, and define it by the relation:

$$I_{peb}(z\,f) = I_{homo}(z). \qquad (74)$$

Introducing this relation with the modified Coveyou formula, we get

$$f = \frac{\rho\lambda\tau}{-\log_e[1-\rho+\rho e^{-\lambda\tau}]} \quad , \qquad (75)$$

where $\lambda = (u + v)^{-1}$

and for $(1 - \rho + \rho e^{-\tau\lambda}) \approx 1$:

$$\lim_{\rho\to 0} f = \frac{\lambda\tau}{1 - e^{-\lambda\tau}} \quad . \qquad (76)$$

We note also that

$$\lim_{\rho\to 1} f = 1, \qquad (77)$$

$$\lim_{\lambda\tau\to 1} f = 1, \qquad (78)$$

$$\lim_{\lambda\tau\to N\gg 1} f = \frac{\rho N}{-\log_e(1-\rho)} \quad . \qquad (79)$$

A graph of f vs $\lambda\tau$ for various values of ρ is given in Figure 7. A useful approximation can be made for the channelling-effect factor which is in agreement with (75) to about 15% in f-1, and can be used for quick estimates of the effect for engineering design purposes until more accuracy is needed:

$$f = 1 + 0.54\,\lambda\tau\,(1-\rho); \quad \lambda\tau < 1. \qquad (75a)$$

10. Diffusion in a Pebble Bed

The interesting question now appears of the effect, if any, of the finite voids on the diffusion constants in a pebble pile.

Figure 7

The channelling effect factor, defined as the ratio for the same attenuation of the thickness of a pebble shield to that of a homogeneous shield of equal average density and identical microscopic properties, is here illustrated according to the interpretation given by the discrete theory. The parameter λ is a constant proportional to the mean chord length of the pebble. The slope of the curves starts out as $\frac{1}{2}(1 - \rho)$ and increases monatonically to $(1 - \rho)$. The approximate straightness in this region of interest ($\lambda\tau \leq 1$) leads to a simple rule of thumb:

$$f = 1 + 0.54\,(1 - \rho)\,\lambda\tau \quad .$$

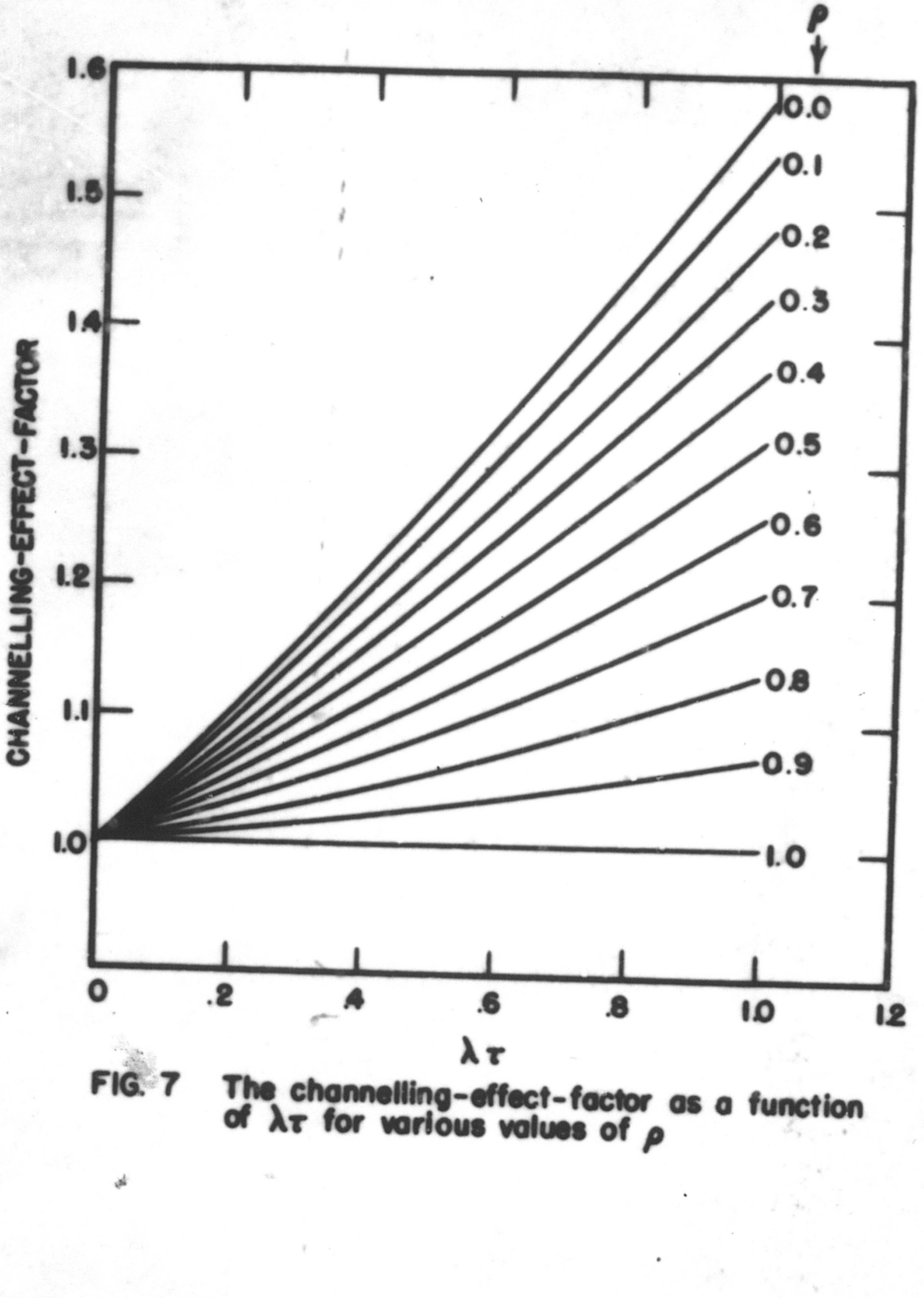


FIG. 7 The channelling-effect-factor as a function of $\lambda\tau$ for various values of ρ

From consideration of diffusion as a random walk[10], we

[10]There is danger in applying the theory of random walk blindly to a heterogeneous system composed of solid and void spaces. This is shown very clearly in computing the first and second moments of the random walk distribution of the case of diffusion perpendicular to alternate solid slabs and void spaces -- all solid and void thicknesses being equal. One gets here that the first moment -- the mean free path -- is that given by the density correction alone; but the second moment (used in the computation of the diffusion length) is increased over and above the density effect.

This is in contradiction to the present theory and to computations made with boundary values as well as being offensive to ones intuitive judgment.

The paradox enters from the assumption implicit in the random walk -- that one is permitted to select any step at random. However, if the step selected ends in a void space it must be rejected as physically unacceptable. Thus the random walk argument cannot be applied.

One might first think that the same criticism applies to the present argument. However, there is a constant probability, due to the statistically isotropic character of the pebble bed, that any step selected must be rejected, and, therefore, the distribution is preserved on the whole, and the random walk argument is applicable.

have that the probability of a step terminating in dV = dx dy dz on the N^{th} step is

$$P(x, y, z)\,dx\,dy\,dz = P(x)\,dx\;P(y)\,dy\;P(z)\,dz$$

$$= \frac{1}{\lambda^3 (2\pi)^{3/2}} \exp \frac{x^2 + y^2 + z^2}{2\lambda^2}\,dx\,dy\,dz \qquad (80)$$

where $\lambda^2 = \frac{1}{3}\overline{l^2}\,N$,

l = length of given step,

N = average number of steps from birth to absorption.

Now the mean square distance between birth and absorption is given by:

$$\overline{r^2} = \int r^2 \, P(x,y,z) \, dV$$

$$= 2\lambda^2 = \overline{l^2} \; N$$

but the diffusion length L^2 is given by

$$L^2 = \frac{\overline{r^2}}{6} = \frac{\overline{l^2} N}{6} \qquad (81)$$

For an exponential scattering law, the probability of path length l is

$$p(l) = \frac{1}{\lambda} e^{-\frac{l}{\lambda}} .$$

One can show easily that the first moment (the mean free path) is

$$\int_0^\infty l \, p(l) \, dl = \lambda$$

and the second moment

$$\overline{l^2} = \int_0^\infty l^2 \, p(l) \, dl = 2\lambda^2 . \qquad (82)$$

Therefore, we have derived that

$$L^2 = \frac{\overline{l^2} N}{6} = \frac{\lambda^2 N}{3} . \qquad (83)$$

Letting $N = \dfrac{\sigma_a + \sigma_s}{\sigma_a}$ equation (41) is shown to be approximately the usual expression for L^2. Now insofar as the channelling effect produces an exponential law of decreased absorption (or scattering) the scattering mean free path is increased by a factor similar to that used in the case of complete absorption on the first collision. However, the magnitude of the effect is changed as one introduces the transport mean free path in place of the absorption relaxation

length and furthermore the mean chord is reduced.[8] Thus, since $e^{-N\sigma x}$ is the transport diffusion kernel for a plane collimated source, one has in these considerations the transport correction to the diffusion in the pebble bed. The channelling effect factor on the iron cylinders described was 1.082 (for relaxation length in iron at 4.8 cm.). For graphite pebbles of similar size and shape the diffusion length will be augmented by a factor of about 1.114 over the density effect, as the mean-free path for a neutron in graphite is ≈ 2.8 cm. Thus the critical mass of a pebble pile made of such pebbles will be increased by a factor of 1.4 over the density correction, or, since $\rho = \frac{1}{2}$, of a total factor of 5.6 over the solid homogeneous pile.

The effect, then, in diffusion is of such magnitude that it must be considered, even in approximate computations.

The channelling effect on the Fermi age will be comparable to but usually less than the channelling effect on the diffusion length, because the scattering mean-free-path is usually longer at the high energies.

The effect herein described is a purely geometrical effect and is independent of the geometry of a particular problem.

10. Random Heterogeneous Mixtures of Two Materials.

One may also raise the question as to what channelling effect exists in a random heterogeneous mixture of two materials -- such as iron pebbles in cement -- , that is, to a situation where the voids are filled with a different material.

This problem may be reduced to the preceding problem (i.e. one of random aggregates of pebbles with voids in the interstices) by a very simple device.

Let the absorption coefficient in the pebbles be τ_1, that of the interstices be τ_2. One then can subtract out of the problem the homogeneous component of the absorption leaving a pseudo solid with absorption coefficient $\tau_1 - \tau_2$ (assuming $\tau_1 > \tau_2$) with interstices of zero absorption. If $\tau_2 > \tau_1$ the situation is reversed. One then computes the channelling effect correction on the "heterogeneous component ", getting its effective attenuation, then adds in the "homogeneous component".

For instance, if the bed of iron pebbles described above having a channelling effect factor of 1.08 were to have its interstices filled with a material having an absorption coefficient of one-half that of the iron, the channelling effect factor would drop to 1.02.

A simple inspection of the distribution equations will justify this procedure analytically.

11. Acknowledgements

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